## Exponential and Logarithmic Functions

### Exponential Functions

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<th>Main Overarching Questions:</th>
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<td>1. Evaluate exponential functions.</td>
<td>1. How do you graph exponential functions?</td>
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<td>2. Graph exponential functions.</td>
<td>2. How do you use compound interest formulas?</td>
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<td>3. Evaluate functions with base e.</td>
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<td>4. Use compound interest formulas.</td>
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<th>Activities and Questions to ask students:</th>
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<tr>
<td>• Evaluate exponential functions</td>
<td>• Define two functions: ( f(x) = 2^x ) and ( g(x) = x^2 ). Explain that the first function is called an exponential function while the 2(^{nd}) function is the previously studied quadratic (or more generally polynomial) function. What is the difference between the two functions?</td>
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<tr>
<td>• Graph exponential functions</td>
<td>• Define the general exponential function. Have students practice evaluating these types of functions for a variety of values. Consider a demonstration with the calculator.</td>
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</table>
| • Have students create a table of values to graph the function \( f(x) = 2^x \). You may have to review negative exponents, but students should use positive and negative values of \( x \). After graphing the function, have a discussion about the properties of the graph. What happens as \( x \) gets very small (negative)? What happens as \( x \) gets very large? Discuss the domain and range of the exponential function. Are there any values of \( x \), we can’t plug in to the function? Can the function \((y)\) value be negative ever? Why? Why not? What happens at \( x = 0 \)?
Logarithmic of Functions

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<td>• Evaluate functions with e.</td>
<td>1. How do you evaluate logarithms?</td>
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<td>• Use compound interest formulas</td>
<td>2. How do you graph logarithmic functions?</td>
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<td>3. How do you find the domain of a logarithmic function?</td>
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Next, have students repeat the process with $g(x) = \left(\frac{1}{2}\right)^x$. The same questions can be asked.

Have students compare the shapes of the graphs where the base is between 0 and 1 and when the base is greater than 1.

Have students practice graphing exponential functions with tables and using the previously used transformations.

Have students compare the shapes of the graphs where the base is between 0 and 1 and when the base is greater than 1.

Have students practice graphing exponential functions with tables and using the previously used transformations.
6. Find the domain of a logarithmic function
7. Use common logarithms
8. Use natural logarithms

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| • Changing from logarithmic to exponential form. | • Define the logarithmic function as the inverse of the exponential function. How would we find the inverse of \( y = b^x \)? Finding the inverse, we would get \( x = b^y \). But how do we now solve for \( y \)? Since there is no way to solve for \( y \), we introduce the logarithm.
| • Define the inverse of \( y = b^x \) as \( y = \log_b x \). \( b \) is called the base of the logarithmic function. | • To get students to understand the idea that \( y = \log_b x \) is equivalent to \( x = b^y \), start with the inverses. Ask students once more to find the inverse of \( y = b^x \). They should repeat that the inverse is \( x = b^y \). But \( y = \log_b x \) is also the inverse to \( y = b^x \), which means \( x = b^y \) and \( y = \log_b x \) are **equivalent**.
| • To get students to understand the idea that \( y = \log_b x \) is equivalent to \( x = b^y \), start with the inverses. Ask students once more to find the inverse of \( y = b^x \). They should repeat that the inverse is \( x = b^y \). But \( y = \log_b x \) is also the inverse to \( y = b^x \), which means \( x = b^y \) and \( y = \log_b x \) are **equivalent**. | • Ask students to compare \( x = b^y \) and \( y = \log_b x \). What is different? How are the \( x \) and \( y \) in a different place in each form? What about the base \( b \)? Have students note the similarities and differences between the forms.
| • Have students practice changing something in logarithmic form to exponential form. | • Have students practice changing something in exponential form to logarithmic form. |
| • Changing from exponential to logarithmic form | • Evaluate logarithms

- To evaluate logarithms sometimes we can use its equivalent exponential form.
- Begin by giving an example like \( y = \log_2 16 \). Have students rewrite in exponential form. In the new form can you guess what \( y \) is? What question did you need to answer in order to find \( y \)? (i.e. \( b^y \) to the what power equals \( 16 \)).
- Give another one like this to try.
- To finish out the concept give a problem in the form \( \log_2 64 = \). Ask students to use the process from above to evaluate the log. Students may ask where the “\( y \)” is.
- Give a few more problems that involve more complicated powers like \( \log_7 \sqrt{7} \) etc.
| Use basic logarithmic properties | We want to explore the two special log properties: $\log_b b = ?$ and $\log_b 1 = ?$
To make things simpler, have students determine $\log_3 3 = ?$ and $\log_8 8 = ?$. Do you notice any patterns? Have students draw the conclusion that $\log_b b = 1$
Repeat this process to show that $\log_b 1 = 0$, $\log_b b = x$, and $b^{\log_b x} = x$

| Graph logarithmic functions | Have students graph $y = 2^x$ by making a table of ordered pairs. Ask students how they would graph $y = \log_2 x$. What is the relationship between the two graphs? Hopefully students will remember they are inverses. How can we use the ordered pairs of $y = 2^x$ to graph $y = \log_2 x$?
Have students graph $y = \log_2 x$.
Have a discussion with the students about the properties of logarithmic graphs. Are there any asymptotes? Is the function increasing or decreasing? Is there an x-intercept? What is the end behavior?
Have students repeat for $y = \log_{\frac{1}{2}} x$
You can repeat the earlier discussion for this function.
Have students compare the shapes of the graphs where the base is between 0 and 1 and when the base is greater than 1.
Have students practice graphing logarithmic functions with tables and using the previously used transformations.

| Find the domain of a logarithmic function | Begin by graphing 3 or 4 logarithmic functions with different expressions within the logarithm. To make things simpler keep the base the same.
Ex: $y = \log_4 x$, $y = \log_4 (x - 2)$, $y = \log_4 (x + 2)$, $y = \log_4 (x - 6)$
In each case, have the students label the domain. Do you notice a pattern? What relationship does the inside of the log have to do with the domain?
Have students practice finding the domain of a logarithmic function.

| Use common logarithms | Define $\log_{10} x = \log x$.
Have students derive the properties for common logarithms from the previously derived properties.

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### Properties of Logarithms

**Overview of Objectives**, students should be able to:

1. Use the product rule
2. Use the quotient rule
3. Use the power rule
4. Expand logarithmic expressions
5. Condense logarithmic expressions
6. Use the change of base property

**Main Overarching Questions:**

1. How do you expand logarithmic expressions?
2. How do you condense logarithmic expressions?
3. How do you use the change of base property to evaluate logarithms?

**Objectives:**

**Activities and Questions to ask students:**

- **Use the product rule**
  - To illustrate the product rule give students a few examples that show the property in use, but allow students to make the connection: \( \log_5 (5 \cdot 6) = \log_5 (5) + \log_5 (6) \).
  - Repeat until students draw the conclusion that \( \log_b (M \cdot N) = \log_b (M) + \log_b (N) \).

- **Use the quotient rule**
  - To illustrate the quotient rule give students a few examples that show the property in use, but allow students to make the connection: \( \log_5 \left( \frac{x}{2} \right) = \log_5 (x) - \log_5 (2) \).
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<td>1. Use like bases to solve exponential equations.</td>
<td>1. How do you solve exponential equations?</td>
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<tr>
<td>2. Use logarithms to solve exponential equations.</td>
<td>2. How do you solve logarithmic equations?</td>
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### Exponential and Logarithmic Equations

#### Overview of Objectives

- Repeat until students draw the conclusion that \( \log_b \left( \frac{M}{N} \right) = \log_b (M) - \log_b (N) \)
- **Use the power rule**
  - To illustrate the power rule give students a few examples that show the property in use, but allow students to make the connection: \( \log_b (x^2) = 2 \log_b (x) \).
  - Repeat until students draw the conclusion that \( \log_b (M^p) = p \log_b (M) \)
  - Before moving on, allow students to practice using the three properties just described.
- **Expand logarithmic expressions**
  - Have students practice expanding logarithmic expressions.
  - Order is a key issue so give a few similar examples and ask students about the differences in the expansion. Ex: \( \log_2 (x^2 y) \) vs. \( \log_2 ((xy)^2) \)
- **Condense logarithmic expressions**
  - Have students practice condensing logarithmic expressions.
- **Use the change of base property**
  - To begin the illustration, if not already done, have students practice using the calculator to evaluate common and natural logarithms.
  - Ask students how they would evaluate a logarithm like \( \log_2 5 \). If they suggest rewriting it in exponential form, have them attempt this. Why does this not work?
  - Introduce the change of base formula: \( \log_b M = \frac{\log_{10} M}{\log_{10} b} \).
  - Ask students how this formula works. Where do you use the base? Where do you use the inside of the logarithm?
  - Have students practice using this formula?

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### Objectives:

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<td>3.</td>
<td>Use the definition of a logarithm to solve logarithmic equations</td>
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<td>4.</td>
<td>Use the one-to-one property of logarithms to solve logarithmic equations</td>
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<tr>
<td>5.</td>
<td>Solve applied problems involving exponential and logarithmic equations</td>
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### Activities and Questions to ask students:

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<tr>
<td>• Use like bases to solve exponential equations.</td>
<td>• Begin by introducing exponential equations. How do you know it is an exponential equation? Where is the variable? (in the exponent)</td>
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<td>• To solve the equations in this subsection, like bases will be used.</td>
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<td>• Start with a very simple example like $3^x = 9$. Ask students to observe the answer. Hopefully they will see that $x = 2$. How do you know?</td>
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<td>• Ask students if it is possible to rewrite the right hand side in terms of the base 3. In other words, can rewrite 9 as a power of 3? 3 to the what power is 9? Summarize the results as $3^x = 3^2$</td>
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<td>• With the preceding equation, do you see how $x = 2$?</td>
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<td>• Repeat with $3^x = 27$</td>
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<td>• Have students write down the procedure to solve exponential equations with like bases.</td>
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<tr>
<td>• Use logarithms to solve exponential equations</td>
<td>• To segue to this next concept, have students try to solve $4^x = 15$. Can 15 be rewritten as a power of 4? Why not?</td>
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<td>• The goal is to get $x$ by itself, but $x$ is in the exponent. Ask student to consider taking the ln of both sides. Assure them that it is permissible by algebra. If this is done we have $\ln(4^x) = \ln(15)$</td>
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<td>• Now, have students consider the properties of logarithms and think about a way to simplify</td>
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the left hand side. This might take some doing, but it’s important to have them realize the equation simplifies to $x \cdot \ln(4) = \ln(15)$
- At this point, some students might not know how to isolate $x$. Ask if ln(4) is number or a variable. What is ln(14)? It’s a number! So how do we move numbers that are multiplied times a variable? Division! Have students finish the problem.
- This is a more complicated procedure so have students summarize the process.
- Examples can very greatly in difficulty, so in class give a variety to have students practice.

<table>
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<tr>
<th>Use the definition of a logarithm to solve logarithmic equations</th>
<th>Introduce a logarithmic equation like $\log_3 x = 5$. What is different about this equation?</th>
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<tr>
<td>Use the one-to-one property of logarithms to solve logarithmic equations</td>
<td>Ask students how they might solve it. Can rewrite the logarithm in an equivalent form? Have students think about earlier properties studied.</td>
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<td>Have students rewrite as $3^5 = x$ and then $243 = x$</td>
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<td></td>
<td>Have students summarize this process of solving logarithmic equations.</td>
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<td></td>
<td>Give students a logarithmic equation with multiple but not all logarithmic terms like $\log_2 (x + 2) - \log_2 (x - 5) = 4$. How can we use properties to simplify this equation to one we have already solved?</td>
</tr>
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- Introduce a second type of logarithmic equation where ALL terms are logarithmic. Begin with a simple equation like $\ln x = \ln 2$. What does $x$ have to be in order to make the sides balance? What about $\log_3 x = \log_3 2$. What is $x$? Do the logarithms matter? Have students conclude that if $\log_b M = \log_b N$ then $M = N$.
- Ask students to compare the original equation with the previous logarithmic equations. How do you know which process to use? What is different about the two types of logarithmic equations?
- Give students a worksheet of exponential and logarithmic equations and have them reference each of the 4 types studied and solve.

- Solve applied problems involving exponential and logarithmic equations

An interesting application from Blitzer’s “College Algebra” introduces the exponential equation $P(x) = 95 - 30\log_3 x$ where $P(x)$ represents the percentage of students who recall important features of a classroom lecture after $x$ days. Have students practice solving word problems. The key here is to determine what to plugin and where and then how to solve.
## Exponential Growth and Decay; Modeling Data

### Overview of Objectives, students should be able to:
1. Model exponential growth and decay

### Main Overarching Questions:
1. How do you find and use an exponential model?

### Objectives:
- Model exponential growth and decay

### Activities and Questions to ask students:
- Introduce the exponential model: \( A = A_0e^{kt} \). Explain what each letter represents. What interest formula is this similar to?
- An application is often the best way to illustrate these types of problems. One such application is carbon 14 dating. Define half-life as the time it takes for half of the original quantity to disintegrate. It can be used to date very old artifacts.
- In order to create our model, we need to figure out \( k \) the growth/decay constant. Word problem: Create a model for the amount of carbon-14 left after \( t \) years using the fact that the half-life of carbon-14 is 5715 years.
- How much carbon-14 is left after 5715 years? Students may say “half.” Half of what? “Half of the original amount.” Well how much was originally there? “I don’t know.” That information is not given. What letter did we represent the original amount with? \( A_0 \).
- So when \( t = 5715 \), \( \frac{1}{2} \) (original amount) = \( \frac{1}{2} A_0 \). Have students plug-in these “values” for \( t \) and \( A \). What happens if we divide both sides by \( A_0 \)? Did we need to know the original amount? Now we have the equation \( \frac{1}{2} = e^{k \cdot 5715} \). What kind of equation is this? How do we solve this kind?
- Have students solve the equation for \( k \). What is the sign? Why is it negative? Does that make sense? Is this a growth or decay problem?
- Have students rewrite the model with \( k \) plugged in.
- To extend the concept, have students predict how old the Dead Sea Scrolls are if in 1947 they...
continued 76% of their original carbon-14 using their model. Which variable are we trying to solve for? What is A?
- Repeat this process for an exponential growth model like population or bacteria growth.

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